

Locus

Model construction and reduced dynamics of a Roberts 10-bar

Technical note · four-bar kinematics, projective assembly, Euler–Lagrange reduction · August 2026

This note records the geometry and the dynamics implemented in the simulator. The mechanism is a planar four-bar with a coupler point, completed by its two Roberts–Chebyshev cognates to a ten-bar chain. Special geometry makes the chain one-degree-of-freedom. All subsequent force and inertia calculations are projections of the nine moving rigid bodies onto that single coordinate.

1. Four-bar geometry

Fix a ground segment O_2O_4 of length d . The input crank O_2A has length ℓ_2 , the coupler AB length ℓ_3 , and the rocker O_4B length ℓ_4 . A coupler point P is attached rigidly to AB by body-fixed coordinates (p, q) :

$$P = A + p\hat{t} + q\hat{n}, \quad \hat{t} = \frac{B - A}{\|B - A\|}, \quad \hat{n} = J\hat{t}, \quad (1)$$

where J is rotation by $+\pi/2$. The configuration of the four-bar is completely determined by the crank angle $\theta_2 = \arg(A - O_2)$ once an assembly sheet is chosen.

Loop closure in complex form (identifying the plane with $\mathbb{C}$) is

$$\ell_2 e^{i\theta_2} + \ell_3 e^{i\theta_3} = d + \ell_4 e^{i\theta_4}. \quad (2)$$

The two unknown angles θ_3, θ_4 are recovered by intersecting the circle of radius ℓ_3 about A with the circle of radius ℓ_4 about O_4 . The two real intersections, when they exist, are the open and crossed assemblies.

1.1 Grashof class and circuits

Write $s \leq \dots \leq \ell$ for the sorted side lengths. The linkage is Grashof when $s + \ell$ is strictly less than the sum of the other two sides, and a change-point when equality holds. The input fully rotates if and only if the set of crank angles at which the two circles meet is the whole circle. Otherwise the admissible θ_2 fall into one or more closed intervals (circuits). Dead-centres occur where the circles are tangent,

$$|A - O_4| \in \{|\ell_3 - \ell_4|, \ell_3 + \ell_4\}, \quad (3)$$

and are the only places at which the two assemblies coincide. On a limited circuit the crank reflects at those locks; on a fully rotating crank the sheet is continued through the lock (fold-through) so that the coupler point keeps a continuous velocity.

2. Projective assembly

Each joint circle is a Euclidean circle and therefore a point-conic. In the affine chart $W = 1$,

(4)

$$C(x, y) = x^2 + y^2 + Dx + Ey + F = 0, \quad D = -2c_x, \quad E = -2c_y, \quad F = \|c\|^2 - r^2.$$

The pencil $C_A - \lambda C_{O_4}$ contains a unique degenerate member, the radical axis

$$\mathcal{L}: (D_A - D_4)x + (E_A - E_4)y + (F_A - F_4) = 0. \quad (5)$$

$\mathcal{L}$ is the common chord of the two circles (a projective line, independent of the quadratic terms). Substituting $\mathcal{L}$ into C_A and restricting to the affine chart is equivalent to dropping a perpendicular from A onto $\mathcal{L}$ and walking a half-chord $\sqrt{r^2 - \delta^2}$ along the line, where δ is the signed distance from A to $\mathcal{L}$. The two feet are the candidate locations of B . When $|\delta| = r$ the chord collapses to a point: a lock.

Continuity of the sheet is not algebraic. Given a previous pose (B^-, P^-) and a predicted point $P^- + V_P$, the simulator selects the intersection nearer that prediction. Velocity hints are dropped at a circuit bounce so the crank reverses on the same sheet instead of chattering across the lock.

3. Roberts–Chebyshev cognates

A planar four-bar with a coupler point P has two cognate four-bars that trace the same coupler curve (Roberts, 1875; Chebyshev). They are built from the live pose by a Cayley diagram of parallelograms and similar triangles. Identifying vectors with complex numbers, the construction used here is

$$A_2 = O_2 + (P - A), \quad B_3 = O_4 + (P - B), \quad (6)$$

$$C_2 = A_2 + (P - A_2) \frac{P - A}{B - A}, \quad C_3 = P + (B_3 - P) \frac{P - A}{B - A}, \quad (7)$$

$$O_6 = C_2 + (C_3 - P). \quad (8)$$

The two new four-bars are $O_2A_2C_2O_6$ and $O_4B_3C_3O_6$. All three coupler triangles ABP , A_2C_2P , B_3C_3P remain similar, and P is common. The point O_6 is *geometrically* stationary — a consequence of the similarity, not an extra pin to the ground. In the simulator it is treated as a floating vertex of the two cognate rockers C_2O_6 and C_3O_6 , not as a grounded pivot. A force applied at O_6 is therefore admitted, but its reduced generalised force vanishes identically whenever $\partial r_{O_6} / \partial \theta_2 = 0$.

Counting without the Roberts identities, the ten-bar (three grounded pivots and nine moving bars, or nine moving bars and a floating O_6) is overconstrained. The identities restore mobility one. Grübler’s formula is not the right count; the reduction below uses that fact and never introduces the extra algebraic constraints.

Body	Segment	Role
1	O_2A	input crank
2	AB	original coupler
3	O_4B	original rocker
4	O_2A_2	left cognate crank
5	A_2C_2	left cognate coupler

Body	Segment	Role
6	C_2O_6	left cognate rocker
7	O_4B_3	right cognate crank
8	B_3C_3	right cognate coupler
9	C_3O_6	right cognate rocker

Table 1. Nine moving rods. Ternary plates ABP , A_2C_2P , B_3C_3P are rigid with their coupler bars; they are not given a second mass.

4. Reduced coordinate and Jacobians

Let $q = \theta_2$ be the sole generalised coordinate (abbreviated θ below). Loop closure plus the Cayley map is a function

$$\Phi(\theta) = \{ r_i(\theta), \alpha_i(\theta) \}_{i=1}^9 \in (SE(2))^9, \quad (9)$$

where r_i is the centre of mass of rod i (the midpoint) and α_i is its absolute angle. Differentiating along the allowed motion,

$$\dot{r}_i = J_{v,i}(\theta) \dot{\theta}, \quad \omega_i = J_{\omega,i}(\theta) \dot{\theta}, \quad J_{v,i} = \frac{\partial r_i}{\partial \theta}, \quad J_{\omega,i} = \frac{\partial \alpha_i}{\partial \theta}. \quad (10)$$

The simulator evaluates these Jacobians by a centred (or one-sided, near a circuit end) difference of step $h = 2.5 \cdot 10^{-3}$ on the same assembly sheet. No constraint Jacobian of a DAE is formed: the constraints have already been solved.

5. Inertia

Each rod is a uniform slender bar of linear density μ , so

$$m_i = \mu \ell_i, \quad I_i = \frac{1}{12} m_i \ell_i^2 \quad (11)$$

about the centre of mass, with the axis perpendicular to the plane. There is no gravity. The reduced mass (a scalar) is the pull-back of the body-space kinetic-energy metric:

$$M(\theta) = \sum_{i=1}^9 \left(m_i \|J_{v,i}\|^2 + I_i J_{\omega,i}^2 \right). \quad (12)$$

Kinetic energy is then simply $T = \frac{1}{2} M(\theta) \dot{\theta}^2$. The derivative $M'(\theta)$ is obtained from the same finite-difference stencil and clamped at $|M'| \leq 6M$ so that a lock, where the coordinate chart becomes singular, cannot explode the Christoffel term.

6. Torsion springs

Up to two torsion springs may be attached, each at one of O_2, A, O_4, O_6 . Write $\varphi_j(\theta)$ for the joint angle of spring j and φ_j^0 for its neutral. The potential and the generalised torque are

$$V_{\text{spr}} = \sum_j \frac{1}{2} k_j (\varphi_j(\theta) - \varphi_j^0)^2, \quad Q_{\text{spr}} = - \sum_j k_j (\varphi_j - \varphi_j^0) \varphi_j', \quad (13)$$

with $\varphi_j' = \partial \varphi_j / \partial \theta$. The two neutrals need not agree. The static equilibrium of the unloaded, undamped system is the stationary point of V_{spr} , $\sum_j k_j (\varphi_j - \varphi_j^0) \varphi_j' = 0$, a compromise between the two springs weighted by stiffness and by geometric advantage.

The joint angles used in the code are $\varphi_{O_2} = \theta_2$, $\varphi_A = \theta_3 = \arg(B - A)$, $\varphi_{O_4} = \theta_4 = \arg(B - O_4)$, and $\varphi_{O_6} = \arg(C_2 - O_6)$. Angle differences are taken in $(-\pi, \pi]$.

7. Joint damping

Every revolute is viscous and frictionless in the Coulomb sense. If $\psi_k(\theta)$ is the relative angle at pin k , Rayleigh's dissipation function is

$$\mathcal{R} = \frac{1}{2} c \sum_k \dot{\psi}_k^2 = \frac{1}{2} c D(\theta) \dot{\theta}^2, \quad D(\theta) = \sum_k (\partial \psi_k / \partial \theta)^2. \quad (14)$$

The pins counted are: ground–crank O_2 , ground–rocker O_4 , crank–coupler A , coupler–rocker B ; and, with cognates, ground– A_2 , ground– B_3 , $A_2 C_2$, $C_2 O_6$, $B_3 C_3$, $C_3 O_6$, the floating pin O_6 (relative rate of $C_2 O_6$ against $C_3 O_6$), and the two coupler-to-cognate pins at P . The reduced damping torque is $-c D(\theta) \dot{\theta}$. Setting $c = 0$ recovers a conservative system.

8. Constant force on a vertex

A world-fixed force $F = F(\cos \alpha, \sin \alpha)$ may act at any of $A, B, P, A_2, C_2, B_3, C_3, O_6$. Because F is constant in both magnitude and direction it is conservative, with potential $-F \cdot r$. Virtual work on the allowed motion gives

$$Q_F = F \cdot \frac{\partial r_\star}{\partial \theta}, \quad V_F = -F \cdot r_\star(\theta). \quad (15)$$

If the chosen vertex is O_6 and the Roberts identities hold exactly, then $\partial r_{O_6} / \partial \theta = 0$ and $Q_F = 0$: the force is carried entirely by the two cognate rockers and does no work on the degree of freedom. That is a property of the geometry, not an omission.

9. Equations of motion

The Lagrangian on the reduced space is $L = \frac{1}{2} M(\theta) \dot{\theta}^2 - V(\theta)$ with $V = V_{\text{spr}} + V_F$. The Euler–Lagrange equation, together with Rayleigh dissipation, is

$$\frac{d}{dt}(M\dot{\theta}) - \frac{1}{2} M' \dot{\theta}^2 = Q_{\text{spr}} + Q_F - c D \dot{\theta}. \quad (16)$$

Expanding the first term recovers the familiar one-dimensional form

$$M(\theta) \ddot{\theta} + \frac{1}{2} M'(\theta) \dot{\theta}^2 + c D(\theta) \dot{\theta} = Q_{\text{spr}}(\theta) + Q_F(\theta).$$

Equation (17). Reduced dynamics of the 10-bar. The left-hand side is inertia plus the quadratic velocity term coming from a configuration-dependent metric, plus viscous damping. The right-hand side is all remaining generalised force.

Equivalently, in Hamiltonian variables $p = M\dot{\theta}$,

$$\dot{\theta} = \frac{p}{M}, \quad \dot{p} = \frac{1}{2} M' \left(\frac{p}{M} \right)^2 + Q_{\text{spr}} + Q_F - c D \frac{p}{M}. \quad (18)$$

The simulator integrates (17) with a two-stage Heun step (explicit RK2) and ten substeps per display frame. After each trial step the crank is reflected if it has left its circuit, and the pose is reconstructed by walking θ in increments of at most 0.05 rad on the current sheet, using position hints only (no velocity prediction) so that dynamics cannot flip the assembly.

10. Energy

Mechanical energy

$$E = T + V_{\text{spr}} + V_F = \frac{1}{2} M \dot{\theta}^2 + \sum_j \frac{1}{2} k_j (\varphi_j - \varphi_j^0)^2 - F \cdot r_\star \quad (19)$$

is conserved when $c = 0$. With damping the dissipation identity is $\dot{E} = -c D \dot{\theta}^2 = -2\mathcal{R}$. The on-screen power $\dot{W}$ is exactly that quantity. Finite-difference noise in M' and the Heun discretisation produce a relative drift of order 10^{-3} over a few seconds at the default step; that is a numerical residual, not a missing force.

11. What is not modelled

The reduction assumes the Roberts identities hold, so the ten-bar never leaves the one-dimensional Roberts variety. There is no joint clearance, no flexibility, no Coulomb friction, and no gravity. Ground is inertial. Masses live on the nine unique structural bars; the three coupler triangles are not triple-counted. A full Newton–Euler DAE with Lagrange multipliers for every pin would compute the same $\ddot{\theta}$ after projection onto $\ker(G)$, and would additionally produce the constraint forces. Those multipliers are not needed for the motion and are not formed.

Locus technical note. Geometry: four-bar loop closure by the radical axis of the joint circles; Roberts–Chebyshev cognates by the Cayley diagram (6)–(8). Dynamics: reduced Euler–Lagrange on θ_2 , equations (12)–(17). Matches the implementation in `src/lib/mechanism/`.